2.3 Spanning Sets and linear Independence.

Consider the Vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

a) Is the Vector $\vec{w} = \begin{bmatrix} 7 \\ -1 \\ 23 \end{bmatrix}$

a linear Combination of $\vec{u}$ and $\vec{v}$?

b) Is the Vector $z = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ a linear combination of $\vec{u}$ and $\vec{v}$..?

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answer:

We need to find constant numbers C_1 and C_2 and d_1 and d_2 such that

$$C_1 \vec{u} + C_2 \vec{v} = \vec{w} \quad (1.1) \quad \text{and} \quad d_1 \vec{u} + d_2 \vec{w} = \vec{z} \quad (1.2)$$

if $\vec{w}$ and $\vec{z}$ are linear combinations of $\vec{u}$ and $\vec{w}$.

The equations (1.1) and (1.2) can also be written as linear System of equations.

In fact,

a) I

$$c_1 \overset{\vec{u}}{\begin{bmatrix}1\\2\\-1\end{bmatrix}} + c_2 \overset{\vec{v}}{\begin{bmatrix}2\\1\\4\end{bmatrix}} = \overset{\vec{w}}{\begin{bmatrix}7\\-1\\23\end{bmatrix}}$$

Equivalent Linear system for the unknowns c_1, c_2.

II
$$\begin{cases} c_1 + 2c_2 = 7 \\ 2c_1 + c_2 = -1 \\ -c_1 + 4c_2 = 23 \end{cases}$$

This linear system has an augmented matrix given by $[A|\vec{W}]$

III
$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -1 \\ -1 & 4 & 23 \end{array}\right] \sim \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -3 & -15 \\ 0 & 6 & 30 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 2 & 7 \\ 0 & -3 & -15 \\ 0 & 0 & 0 \end{array}\right)$$

From here, we obtain (Gauss elimination)

$$-3c_2 = -15 \rightarrow \boxed{c_2 = 5}$$

$$c_1 = 7 - 2c_2 = 7 - 2(5) = -3, \boxed{c_1 = -3}$$

b) Similarly, Are there constant coefficients c_1 and c_2 such that vector equation (2.1) is satisfied.

I
$$c_1 \begin{bmatrix}1\\2\\-1\end{bmatrix} + c_2 \begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}3\\6\\-1\end{bmatrix} \qquad (2.1)$$

Also, (2.1) is equivalent to

II
$$\begin{cases} c_1 + 2c_2 = 3 \\ 2c_1 + c_2 = 6 \\ -c_1 + 4c_2 = -1 \end{cases}$$

which can be written as

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & 6 \\ -1 & 4 & -1 \end{array}\right) = [A|\vec{z}]$$

III

Implementing the row operations to obtain an echelon form from the previous augmented matrix We obtain

$$\text{(III)} \quad \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & 6 \\ -1 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & 0 \\ 0 & 6 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{array}\right)$$

Clearly, from last row form there is no solution.

So, our answer to the given problem is

$\vec{w} = \begin{bmatrix} 7 \\ -1 \\ 23 \end{bmatrix}$ is a linear combination of $\vec{u}$ and $\vec{v}$

but $\vec{z} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ is not.

Def.- For $S = \{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ in $\mathbb{R}^n$, the set of all linear combinations

$$c_1\vec{u}_1 + \cdots + c_k\vec{u}_k, \qquad c_1, \ldots, c_k \text{ in } \mathbb{R}$$

is called the span of $\vec{u}_1, \ldots, \vec{u}_k$.

It is denoted as

$$\text{Span}(\vec{u}_1, \ldots, \vec{u}_k) \text{ or } \text{span}(S)$$

and S is called the spanning set of $\text{Span}(\vec{u}_1, \ldots, \vec{u}_k)$.

In our example,

$$\vec{w} = \begin{bmatrix} 7 \\ -1 \\ 23 \end{bmatrix} \text{ is in Span}(\vec{u}_1, \vec{u}_2)$$

but $\vec{z} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ is not.

From our previous example, we learn that a vector $\vec{w}$ is in span $(\vec{u}_1, \vec{u}_2)$ if and only if the linear system with augmented matrix $[A|\vec{w}]$, where the columns of A are $\vec{u}_1$ and $\vec{u}_2$ has a solution.

This is true in general,

Theorem 2.4

$\vec{w}$ is in Span $(\vec{u}_1, \vec{u}_2, .., u_k)$ if and only if the linear system with augmented matrix $[A|\vec{w}]$, where the columns of A are $\vec{u}_1, ., \vec{u}_k$ is Consistent.

2.3 #11)

$$\mathrm{Span}\left(\overset{\vec{u}}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \overset{\vec{v}}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{\vec{w}}{\begin{bmatrix}0\\1\\1\end{bmatrix}}\right) \overset{?}{=} \mathbb{R}^3$$

This is equivalent to say if given an arbitrary vector $\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$ in $\mathbb{R}^3$

if there is a linear combination of $\vec{u}, \vec{v}, \vec{w}$ such that

$$\text{(I)} \qquad c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

This is equivalent to the linear system

$$\text{(II)} \qquad \begin{aligned} c_1 + c_2 &= \alpha \\ c_2 + c_3 &= \beta \\ c_1 + c_3 &= \gamma \end{aligned}$$

which has the augmented matrix

$$\text{(III)} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & \alpha \\ 0 & 1 & 1 & \beta \\ 1 & 0 & 1 & \gamma \end{array}\right)$$

Applying <u>Gaussian Elimination</u>, we can obtain c_1, c_2, and c_3 if they exist. It means if the linear system has solution for an arbitrary $\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$ in $\mathbb{R}^3$.

<u>1st step</u> : <u>Row reduction</u>

$$\left(\begin{array}{ccc|c}1&1&0&\alpha\\0&1&1&\beta\\1&0&1&\gamma\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&0&\alpha\\0&1&1&\beta\\0&-1&1&\gamma-\alpha\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&1&0&\alpha\\0&1&1&\beta\\0&0&2&\gamma-\alpha+\beta\end{array}\right)\sim\left(\begin{array}{ccc|c}1&0&-1&\alpha-\beta\\0&1&1&\beta\\0&0&2&\gamma-\alpha+\beta\end{array}\right)$$

The conclusion from here is that the syst. is consistent for any $\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$ so $\text{Span}(\vec{u},\vec{v},\vec{w}) = \mathbb{R}^3$.

$$\begin{pmatrix}1&0&-1&\alpha-\beta\\0&1&1&\beta\\0&0&1&\frac{\gamma-\alpha+\beta}{2}\end{pmatrix}\sim\begin{pmatrix}1&0&0&\alpha-\beta+\frac{\gamma-\alpha+\beta}{2}\\0&1&0&\beta-\left(\frac{\gamma-\alpha+\beta}{2}\right)\\0&0&1&(\gamma-\alpha+\beta)/2\end{pmatrix}$$

$2^{\underline{nd}}$ step <u>Gauss-Jordan</u>

From the last matrix, we get

$$c_1 = \frac{\alpha}{2} - \frac{1}{2}\beta + \frac{\gamma}{2} \qquad (11.1)$$

$$c_2 = \beta/2 - \gamma/2 + \alpha/2 \qquad (11.2)$$

$$c_3 = \gamma/2 - \alpha/2 + \beta/2 \qquad (11.3)$$

<u>Checking</u>: Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\text{Span}(\vec{u}, \vec{v}, \vec{w})$?

Using the previous expresions for c_1, c_2, c_3

$$c_1 = \frac{1}{2} - \frac{1}{2}(2) + \frac{3}{2} = 2 - 1 = 1$$

$$c_2 = 1 - 3/2 + 1/2 = 0$$

$$c_3 = 3/2 - 1/2 + 1 = 2$$

In fact,

$$1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

A more interesting one

Is $\vec{z} = \begin{bmatrix} \pi \\ \sqrt{2} \\ e \end{bmatrix}$ in Span $(\vec{u}, \vec{v}, \vec{w})$?

Using (11.1)–(11.3), we find $\vec{z} = \begin{bmatrix} \pi \\ \sqrt{2} \\ e \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$$C_1 = \frac{\pi}{2} - \frac{\sqrt{2}}{2} + \frac{e}{2}$$

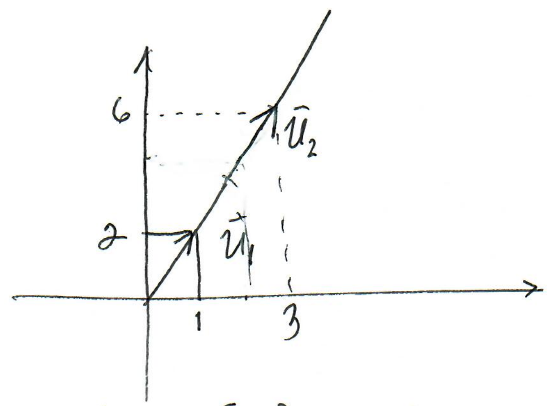
$$C_2 = \frac{\sqrt{2}}{2} - \frac{e}{2} + \frac{\pi}{2}, \qquad C_3 = \frac{e}{2} - \frac{\pi}{2} + \frac{\sqrt{2}}{2}$$

It should be easy to check that for this C_1, C_2 and C_3

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi \\ \sqrt{2} \\ e \end{bmatrix}.$$

LINEAR INDEPENDENCE.

I) Consider the vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ in $\mathbb{R}^2$



Since $\vec{u}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\,\vec{u}_1$

then $\boxed{3\vec{u}_1 - \vec{u}_2 = \vec{0}}$

It means there is a linear combination of $\vec{u}_1$ and $\vec{u}_2$

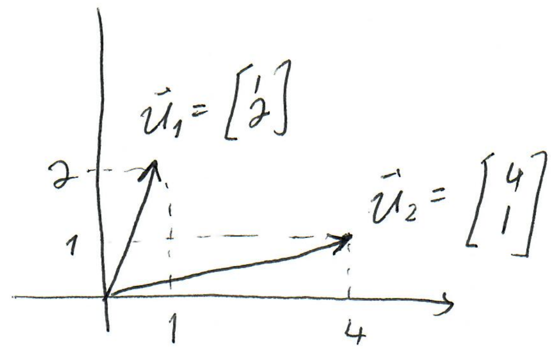
$$c_1\vec{u}_1 + c_2\vec{u}_2 = \vec{0},$$

where c_1 or c_2 are not zero. In this particular case $\boxed{c_1 = 3,}$ $\boxed{c_2 = -1}$

For this example, it will appropriate to say that $\vec{u}_2$ is dependent on $\vec{u}_1$, because it's a multiple of it.

II) Now Consider the Vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



and we also want to explore if their linear Combination

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2.1)$$

Can have c_1 or $c_2 \neq 0$, which in turn means that they are multiples of each other.

Now, (2.1) is equivalent to

$$\begin{aligned} c_1 + 4c_2 &= 0 \\ 2c_1 + c_2 &= 0 \end{aligned}$$

Whose augmented matrix is given by

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 1 & 0 \end{array}\right) \underset{\text{row reduct.}}{\sim} \left(\begin{array}{ccc} 1 & 4 & 0 \\ 0 & -7 & 0 \end{array}\right)$$

Backward Subst.:

$$-7C_2 = 0 \longrightarrow \boxed{C_2 = 0}$$

$$C_1 = -4C_2 = 0, \quad \boxed{C_1 = 0}$$

Therefore, for (2.1)

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only possibility is $\underline{C_1 = 0, \ C_2 = 0.}$

In this example (from graph and analysis), it will be appropriate to say that $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are linearly independent.

<u>Definition.</u> A Set of Vectors $\vec{v}_1, .., \vec{v}_k$ are linearly dependent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \qquad (6.1)$$

and not all coefficients $c_1, .., c_k$ are zero.

On the contrary, if

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

implies that $c_1 = 0,\ c_2 = 0, .., c_k = 0$ (only possibility)

then, the set $\vec{v}_1 .., \vec{v}_k$ is linearly independent.

<u>Theorem 2.5</u> The set $\vec{v}_1 \ldots, \vec{v}_k$ is lin. dep. if and only if one of them can be written as a linear combination of the others.

<u>Proff.</u> $(\rightarrow)$

If $c_j \neq 0$ in (6.1) $1 \leq j \leq k$

then $$\vec{v}_j = \frac{c_1}{c_j}\vec{v}_1 + \cdots + \frac{c_{j-1}}{c_j}\vec{v}_{j-1} + \frac{c_{j+1}}{c_j}\vec{v}_{j+1} + \cdots \frac{c_k}{c_j}\vec{v}_k$$

$(\leftarrow)$ If $v_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_k\vec{v}_k$

$$\Rightarrow \quad c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_k\vec{v}_k - \vec{v}_i = \vec{0}$$

Therefore, they are lin. dep.

Theorem 2.6 1) $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$

2)
$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix}$$

Then

$\vec{v}_1, \dots, \vec{v}_m$ are lin. dep. if and only if the linear system with augmented matrix $[A \mid \vec{0}]$ has nontrivial soln.

Proof. It's a direct consequence of the equivalence between the two problems:

i) Find $c_1, \dots, c_m$ such that
$$c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = \vec{0}$$

and

ii) Find the soln vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ that solves the homogeneous linear system with augmented matrix $[A \mid \vec{0}]$, where

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \\ \downarrow & & \downarrow \end{bmatrix}$$

Linear Independence

2.3. #28) Given

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

We want to determine if they are linearly independent or not, which means if all C's are zero when

$$\boxed{C_1 \vec{u}_1 + C_2 \vec{u}_2 + C_3 \vec{u}_3 = \vec{0}}$$

I or

$$C_1 \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix} + C_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to

II

$$\begin{aligned} -C_1 + 3C_2 + 2C_3 &= 0 \\ C_1 + 2C_2 + 3C_3 &= 0 \\ 2C_1 + 2C_2 + C_3 &= 0 \\ C_1 + 4C_2 - C_3 &= 0 \end{aligned}$$

Homogeneous System

Def A linear syst. is homogeneous if the constant term in each equation is zero.

and whose augmented matrix is

III

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 4 & -1 & 0 \end{array}\right) \begin{array}{c} (-R_1) \\ \\ \\ \\ \end{array} = [A \mid \vec{0}]$$

Gauss Elimination

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 4 & -1 & 0 \end{array}\right) \xrightarrow{\substack{(-1)R_1+R_2=R_2,\ (-1)R_1+R_4=R_4 \\ (-2)R_1+R_3=R_3}} \sim \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 7 & 1 & 0 \end{array}\right) \underset{\sim}{(1/5)R_2=R_2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 7 & 1 & 0 \end{array}\right) \xrightarrow{(-1)R_2+R_3=R_3,\ (-7)R_2+R_4=R_4} \sim \left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \underset{\sim}{\frac{1}{-3}R_3=R_3}$$

$$\sim \begin{pmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 \end{pmatrix} \underset{\sim}{R_3(-6)+R_4=R_4} \begin{pmatrix} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2^{nd} step <u>Backward Substitution:</u>

$c_3 = 0$, $c_2 = -c_3 = 0.$, $c_1 = 2c_3 + 3c_2$
or $c_1 = 2(0) + 3(0) = 0$

So all C's are zero and the Set $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ is linearly independent.

<u>Thm 2.7</u> 1) $\vec{V}_1, \vec{V}_2, .., \vec{V}_m$ row vectors in $\mathbb{R}^n$

2) $$A = \begin{bmatrix} \vec{V}_1 \longrightarrow \\ \vec{V}_2 \longrightarrow \\ \vdots \\ \vec{V}_m \longrightarrow \end{bmatrix}$$ $m \times n$ with rows $\vec{V}_1, .., \vec{V}_m$.

Then $\vec{V}_1, .., \vec{V}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.

<u>Thm 2.8</u> Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

<u>Proof:</u> Let $\vec{V}_j = \begin{bmatrix} (V_j)_1 \\ \vdots \\ (V_j)_n \end{bmatrix}$

Using thm 2.6, what **we** need to prove is that the homog. linear system with augmented matrix

$$\begin{array}{cccc} \vec{V}_1 & \cdots & \vec{V}_m & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} (V_1)_1 & \cdots & (V_m)_1 & 0 \\ (V_1)_2 & & (V_m)_2 & 0 \\ \vdots & & \vdots & \vdots \\ (V_1)_n & & (V_m)_n & 0 \end{array}\right]$$

has a nontrivial Soln.
Now, if $m > n$, this matrix has more rows than columns.
Therefore, after row-reducing it at least $m-n$ free Variables will appear. This implies that the homog. syst has nontrivial Solns.

Thm 2.7 1) $\vec{V}_1, \ldots, \vec{V}_m$ row vectors in $\mathbb{R}^n$

2) $$A = \begin{pmatrix} \vec{V}_1 \longrightarrow \\ \vec{V}_2 \longrightarrow \\ \vdots \\ \vec{V}_m \longrightarrow \end{pmatrix}_{m \times n} \begin{array}{l} R_1 \\ R_2 \\ \vdots \\ R_m \end{array}$$

Then, $\vec{V}_1, \vec{V}_2 \ldots, \vec{V}_m$ are linearly dependent if and only if $\operatorname{rank}(A) < m$.

Proof. $(\rightarrow)$ $\vec{V}_m = C_1 \vec{V}_1 + \ldots + C_{m-1} \vec{V}_{m-1}$

Then,

row operation: $R_m - C_1 \vec{V}_1 \rightarrow R_m = C_2 \vec{V}_2 + \cdots + C_{m-1} \vec{V}_{m-1}$

row oper: $R_m - C_2 \vec{V}_2 \rightarrow R_m = C_3 \vec{V}_3 + \cdots + C_{m-1} \vec{V}_{m-1}$

$\vdots$

row oper: $R_m - C_{m-1} \vec{V}_{m-1} \rightarrow R_m = \vec{0}$

So $R_m - C_1 \vec{V}_1 - C_2 \vec{V}_2 + \cdots - C_{m-1} \vec{V}_{m-1} = \vec{0}$.

Then $\operatorname{rank}(A) < m$, because the last row of one of its echelon form is zero vector.

$(\leftarrow)$

If $\operatorname{rank}(A) < m$, that means that after several row operations on matrix A a row of zeros is produced. Following this sequence of Operations, we find a linear(*) combination of the original vectors $\vec{V}_1, \ldots, \vec{V}_m$ equal to the zero vector.

turn